

Analysis of hierarchical SSOR for three dimensional isotropic model problem

Pawan Kumar¹

Department of computer science

KU Leuven

Leuven, Belgium

pawan.kumar@cs.kuleuven.be

Abstract

In this paper, we study a hierarchical SSOR (HSSOR) method which could be used as a standalone method or as a smoother for a two-grid method. It is found that the method leads to faster convergence compared to more costly incomplete LU (ILU(0)) with no fill-in, the SSOR, and the Block SSOR method. Moreover, for a two-grid method, numerical experiments suggests that HSSOR can be a better replacement for SSOR smoother both having no storage requirements and have no construction costs. Using Fourier analysis, expressions for the eigenvalues and the condition number of HSSOR preconditioned problem is derived for the three-dimensional isotropic model problem.

Keywords: Fourier analysis, iterative method, preconditioners, eigenvalues, eigenvectors, GMRES

AMS classification: 65F10, 65F15, 65F08, 65T99

1 Introduction

Fourier analysis has been an indispensable tool in understanding existing algorithms and designing newer algorithms in computational science. Similar techniques have been used extensively for analyzing the iterative methods for solving linear systems of the form,

$$Ax = b \tag{1}$$

which lies at the heart of plenty of scientific simulations ranging from computational fluid dynamics to sparse numerical optimization.

The matrix A may be ill-conditioned and some preconditioning is often necessary during an iterative procedure. A possible preconditioned linear system is a transformation of the linear system (1) to $B^{-1}Ax = B^{-1}b$ where B is an approximation to A . The basic linear iteration for solving the preconditioned linear system (1) above is given as follows

$$x^{n+1} = x^n + B^{-1}(b - Ax^n).$$

The iteration is stopped as soon as the residual $b - Ax^n$ is small enough under suitable norm. However, the fixed point iteration shown above is very slow, and it is usually replaced by a more sophisticated and relatively robust Krylov subspace based methods [12] where the solution is improved with the help of Krylov space $\{b, B^{-1}Ab, \dots, (B^{-1}A)^r b\}$. When the preconditioned operator $B^{-1}A$ is symmetric positive definite, a popular choice is the conjugate gradient method [12]. The convergence of the conjugate gradient method depends on the condition number of $B^{-1}A$ which in this case is simply the ratio of its largest and smallest eigenvalue.

¹Some part of this work was done when the author was supported by Fonds de la recherche scientifique (FNRS)(Ref: 2011/V 6/5/004-IB/CS-15) and the renewed contract (Ref: 2011/V 6/5/004-IB/CS-9980) at Université Libre de Brussels, Belgique and the remaining work was done at KU Leuven, Leuven, Belgium

Most of the properties such as eigenvalue bounds and the condition number of the classical preconditioners including Jacobi, Gauss-Siedel, SSOR, alternating direction method (ADI), sparse incomplete LU with no fill, i.e., ILU(0), and modified ILU (MILU) which were earlier obtained by difficult analysis, were obtained easily and elegantly via Fourier analysis by Chan and Elman in [2]. For a two dimensional model problem, Fourier analysis was used to determine the condition number of block MILU for a hyperbolic model problem [10]. Using a similar approach, the condition number of ILU(0) and MILU for a three dimensional anisotropic model problem is derived in [4]. On the other hand, for two-grid and multigrid methods, Fourier analysis has been used to understand the action of the smoother in damping the error [14]. Recently, the author used similar analysis to determine the condition number of a filtering preconditioner [8].

A preconditioner known as RNF(0,0) was first introduced in [7]. It is a modified form of the nested factorization method introduced in [1]. For simplicity and its resemblance to SSOR method, we shall call this method *Hierarchical* SSOR (HSSOR). We call it hierarchical because the preconditioner is built hierarchically using the SSOR preconditioner on lower dimensions thereby using the structure of the matrix. Like point-wise SSOR method, the HSSOR method has no storage requirements and has no construction cost. On the other hand, unlike block SSOR method where explicit factorization of 2D blocks are required, for HSSOR, no explicit factorization of any lower dimensional block is ever performed. In fact, the 2D plane blocks themselves have SSOR factors. The convergence of the method is faster compared to ILU(0) as shown in this paper. Our empirical results suggests that HSSOR is a better replacement for SSOR (or Gauss-Siedel) smoother which is widely employed in the two-grid schemes. For symmetric positive definite coefficient matrix, it was proved in [7] that the HSSOR method (called RNF(0,0) in [7]) is convergent. In this paper, we derive the precise expression for eigenvalue and the condition number of the HSSOR preconditioned matrix for the three-dimensional isotropic model problem. Since our analysis uses the same model problem used in [4], our condition number estimate can be compared to that of ILU(0).

The rest of this paper is organized as follows. In section 2, we introduce some notations, the model problem that we use, and finally we discuss the HSSOR method. In section 3, the Fourier eigenvalues and eigenvectors are derived, and several properties including the condition number estimate is presented. The numerical experiments and comparisons are presented in section 4. Finally, section 5 concludes the paper.

2 Model problem and the preconditioner

The model problem is the following three-dimensional anisotropic equation:

$$-(l_1 u_{xx} + l_2 u_{yy} + l_3 u_{zz}) = r \quad (2)$$

defined on a unit cube $\Omega = \{0 \leq x, y, z \leq 1\}$, with $l_1, l_2, l_3 \geq 0$, and with the periodic boundary conditions as follows

$$u(x, y, 0) = u(x, y, 1), \quad u(x, 0, z) = u(x, 1, z), \quad \text{and} \quad u(0, y, z) = u(1, y, z). \quad (3)$$

The discretization scheme considered in the interior of the domain is the second order finite differences on a uniform $n \times n \times n$ grid, with mesh size $h = 1/(n+1)$ along x , y , and z directions. Here we shall use the notation h to denote the mesh size for the periodic case. With this discretization, we get a system of equation

$$Au = b. \quad (4)$$

It is useful to express the matrix A arising from the periodic boundary conditions using the notation of circulant matrices and the Kronecker products. We introduce these notations as follows.

Definition 1. Let C be a matrix of size $pq \times pq$. We call C a **block circulant** matrix if it has the following form

$$C = Bcirc_p(C_0, C_{p-1}, \dots, C_2, C_1) = \begin{pmatrix} C_0 & C_{p-1} & \cdots & C_2 & C_1 \\ C_1 & C_0 & C_{p-1} & \ddots & \vdots \\ \vdots & C_1 & C_0 & \ddots & \vdots \\ C_{p-2} & & \ddots & \ddots & C_{p-1} \\ C_{p-1} & C_{p-2} & \cdots & C_1 & C_0 \end{pmatrix}_{pq \times pq},$$

where each of the blocks C_i are matrices of size $q \times q$ each. We observe that a block circulant matrix is completely specified by a block row. However if $q = 1$, then we call it **circulant matrix** and denote it by $circ_p(C_0, C_{p-1}, \dots, C_2, C_1)$.

Notation 1. Further, for **block circulant tridiagonal** matrices we introduce the following notation

$$Bctrid_p(C_2, C_0, C_1) = \begin{pmatrix} C_0 & C_1 & & & C_2 \\ C_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & C_0 & C_1 \\ C_1 & & & C_2 & C_0 \end{pmatrix}_{pq \times pq},$$

where each of the blocks C_i are matrices of size $q \times q$ each. However if $q = 1$, then we denote it by $ctrid_p(C_2, C_0, C_1)$.

Notation 2. For **block tridiagonal matrix** with constant block bands we introduce the following notation

$$Btrid_p(F_2, F_0, F_1) = \begin{pmatrix} F_0 & F_1 & & & \\ F_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & F_0 & F_1 \\ & & & F_2 & F_0 \end{pmatrix}_{pq \times pq},$$

where each of the blocks F_i are matrices of size $q \times q$ each. If $q = 1$, then we denote it by $trid_p(F_2, F_0, F_1)$.

Definition 2. The Kronecker product \otimes is an operation on two matrices of arbitrary size resulting in a block matrix. Let $A = (a_{i,j})$ and $B = (b_{i,j})$, then by $A \otimes B$ we mean

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \ddots & & \vdots \\ \vdots & \cdots & & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}.$$

If the difference operators are scaled by step size h^2 , then equation of (4) corresponding to the $(i, j, k)^{th}$ grid point is the following:

$$\begin{aligned} & a_{i,j,k}u_{i,j,k} + b_{i,j,k}u_{i+1,j,k} + c_{i,j,k}u_{i,j+1,k} + d_{i,j,k}u_{i-1,j,k} \\ & + e_{i,j,k}u_{i,j-1,k} + f_{i,j,k}u_{i,j,k+1} + g_{i,j,k}u_{i,j,k-1} = w_{i,j,k}, \end{aligned} \quad (5)$$

where $1 \leq i, j, k \leq n$, and

$$\begin{aligned} & b_{i,j,k} = 0, \quad i = n, \quad c_{i,j,k} = 0, \quad j = n, \\ & f_{i,j,k} = 0, \quad k = n, \quad d_{i,j,k} = 0, \quad i = 1, \\ & e_{i,j,k} = 0, \quad j = 1, \quad g_{i,j,k} = 0, \quad k = 1. \end{aligned} \quad (6)$$

For an anisotropic model problem, we have $a_{i,j,k} = 2(l_1 + l_2 + l_3)$, $b_{i,j,k} = -l_1$, $c_{i,j,k} = -l_2$, $d_{i,j,k} = -l_1$, $e_{i,j,k} = -l_2$, $f_{i,j,k} = -l_3$, $g_{i,j,k} = -l_3$, where $w_{i,j,k} = h^2 r(i, j, k)$. Here the subscript (i, j, k) corresponds to the grid location (ih, jh, kh) . Let I_k denote the identity matrix of size $k \times k$. Using the notation of circulant matrix and Kronecker product, the coefficient matrix corresponding to formula (5) is expressed as follows

$$A = Bctr_{id_n}(-l_3 I_{n^2}, \widehat{D}, -l_3 I_{n^2}), \quad \widehat{D} = Bctr_{id_n}(-l_2 I_n, \overline{D}, -l_2 I_n), \quad \overline{D} = ctr_{id_n}(-l_1, d, -l_1).$$

We consider now the same problem (21) with the following Dirichlet boundary condition

$$u(x, y, 0) = 0, \quad u(x, 0, z) = 0, \quad u(0, y, z) = 0. \quad (7)$$

To differentiate the Dirichlet problem with that of periodic problem, we shall use bold face letters to denote the matrices corresponding to the Dirichlet case. Using second order finite differences with the Dirichlet boundary conditions (7) above, we obtain the matrix \mathbf{A} corresponding to the Dirichlet case as follows

$$\mathbf{A} = \mathbf{D} + \mathbf{L}_1 + \mathbf{L}_1^T + \mathbf{L}_2 + \mathbf{L}_2^T + \mathbf{L}_3 + \mathbf{L}_3^T,$$

where

$$\mathbf{L}_3 = Btr_{id_n}(-l_3 I_{n^2}, 0, 0), \quad \mathbf{L}_2 = I_n \otimes Btr_{id_n}(-l_2 I_n, 0, 0), \quad \mathbf{L}_1 = I_{n^2} \otimes tr_{id_n}(-l_1, 0, 0).$$

For the above model problem the HSSOR preconditioner \mathbf{B} for the Dirichlet problem is defined as follows:

$$\begin{aligned} \mathbf{B} &= (\mathbf{P} + \mathbf{L}_3) (\mathbf{I} + \mathbf{P}^{-1} \mathbf{L}_3^T), \\ \mathbf{P} &= (\mathbf{T} + \mathbf{L}_2) (\mathbf{I} + \mathbf{T}^{-1} \mathbf{L}_2^T), \\ \mathbf{T} &= (\mathbf{M} + \mathbf{L}_1) (\mathbf{I} + \mathbf{M}^{-1} \mathbf{L}_1^T), \end{aligned} \quad (8)$$

where $\mathbf{M} = \text{diag}(\mathbf{A})$. The HSSOR preconditioner defined above is named RNF(0,0) preconditioner in [7]. Using the notation of circulant matrix and the Kronecker product, the HSSOR preconditioner for the periodic boundary condition is now defined as follows

$$\begin{aligned} B &= (P + L_3)(I + P^{-1} L_3^T), \quad P \text{ is of size } n^3 \times n^3, \\ L_3 &= Bcirc_n(0, \dots, 0, -l_3 I_{n^2}), \\ L_3^T &= Bcirc_n(0, -l_3 I_{n^2}, 0, \dots, 0), \\ P &= I_n \otimes P_0, \quad P_0 \text{ is of size } n^2 \times n^2, \\ P_0 &= (\widehat{T} + \widehat{L}_2)(I + \widehat{T}^{-1} \widehat{L}_2^T), \\ \widehat{L}_2 &= Bcirc_n(0, \dots, 0, -l_2 I_n), \\ \widehat{L}_2^T &= Bcirc_n(0, -l_2 I_n, 0, \dots, 0), \\ \widehat{T} &= I_n \otimes T_0, \quad T_0 \text{ is of size } n \times n, \\ T_0 &= circ_n(m + l_1^2/m, -l_1, 0, \dots, 0, -l_1), \\ m &= d. \end{aligned} \quad (9)$$

It can be proved that B is a circulant matrix.

3 Fourier analysis of HSSOR

In this section, we derive the Fourier eigenvalues of the HSSOR preconditioned matrix. For clarity and simplicity, we restrict our analysis to the isotropic problem ($l_1 = l_2 = l_3 = 1$), however, similar analysis holds for the general anisotropic case. In the following, we outline certain assumptions on which our analysis will be based. These assumptions are similar to those made in [2], and has been justified their appropriately.

Firstly, our analysis is for the HSSOR preconditioner B and the coefficient matrix A corresponding to the periodic boundary conditions. Secondly, Fourier analysis is an exact analysis only for constant coefficient matrix, which is indeed the case when we have an isotropic model. According to an argument in [2], the extreme eigenvalues for the periodic and the corresponding Dirichlet problems are same provided $n = 2n_d$. Here $n_d + 1 = 1/h_d$ and h_d is the mesh size for the Dirichlet problem.

Eigenvectors of A are found by applying the operator A to eigenvectors $v^{s,t,r}$. The $(i, j, k)^{th}$ grid component of eigenvector $v^{s,t,r}$ is given by

$$v_{i,j,k}^{s,t,r} = e^{\iota i \theta_s} e^{\iota j \phi_t} e^{\iota k \xi_r}, \quad (10)$$

where $\iota = \sqrt{-1}$, $\theta_s = \frac{2\pi}{n+1}s$, $\phi_t = \frac{2\pi}{n+1}t$, and $\xi_r = \frac{2\pi}{n+1}r$, for $r, s, t = 1, \dots, n$. The eigenvalues $\lambda_{s,t,r}(A)$ of the matrix A is determined by substituting (10) for $u_{i,j,k}$ in the left hand side of (5), and it is found to be

$$\lambda_{s,t,r}(A) = 4 \left(l_1 \sin^2 \frac{\theta_s}{2} + l_2 \sin^2 \frac{\phi_t}{2} + l_3 \sin^2 \frac{\xi_r}{2} \right). \quad (11)$$

For circulant matrices following results hold.

Lemma 1 ([3]). *Any circulant matrix of size n share the same set of eigenvectors.*

Using lemma (1) above, we have the following result.

Lemma 2. *Let S and R be two given circulant matrices with eigenvalues $\lambda_{s,t,r}(S)$ and $\lambda_{s,t,r}(R)$ respectively. Then the eigenvalues of $S + R$ and SR corresponding to the $(s, t, r)^{th}$ grid point is given as follows:*

1. $\lambda_{s,t,r}(S + R) = \lambda_{s,t,r}(S) + \lambda_{s,t,r}(R)$.
2. $\lambda_{s,t,r}(SR) = \lambda_{s,t,r}(S)\lambda_{s,t,r}(R)$.

Proof. It follows easily using lemma (1) above. □

Using the lemma 2 above, the eigenvalues $\lambda_{s,t,r}(B^{-1}A)$ of HSSOR preconditioned matrix is then given by

$$\lambda_{s,t,r}(B^{-1}A) = \frac{\lambda_{s,t,r}(A)}{\lambda_{s,t,r}(B)}, \quad (12)$$

where $\lambda_{s,t,r}(B)$ is given hierarchically as follows:

$$\begin{aligned} \lambda_{s,t,r}(B) &= (\lambda_{s,t,r}(P) + \lambda_{s,t,r}(L_3)) \left(1 + \frac{\lambda_{s,t,r}(L_3^T)}{\lambda_{s,t,r}(P)} \right), \\ \lambda_{s,t,r}(P) &= (\lambda_{s,t,r}(T) + \lambda_{s,t,r}(L_2)) \left(1 + \frac{\lambda_{s,t,r}(L_2^T)}{\lambda_{s,t,r}(T)} \right), \\ \lambda_{s,t,r}(T) &= \lambda_{s,t,r}(M) + \lambda_{s,t,r}(L_1) + \lambda_{s,t,r}(L_1^T), \\ \lambda_{s,t,r}(L_1) &= -l_1 e^{\iota \theta_s}, \quad \lambda_{s,t,r}(L_1^T) = -l_1 e^{-\iota \theta_s}, \\ \lambda_{s,t,r}(L_2) &= -l_2 e^{\iota \phi_t}, \quad \lambda_{s,t,r}(L_2^T) = -l_2 e^{-\iota \phi_t}, \\ \lambda_{s,t,r}(L_3) &= -l_3 e^{\iota \xi_r}, \quad \lambda_{s,t,r}(L_3^T) = -l_3 e^{-\iota \xi_r}. \end{aligned} \quad (13)$$

where $\lambda_{s,t,r}(M) = 6$. The eigenvalues for the matrices $L_1, L_2, L_3, U_1, U_2, U_3$, and M were found by inspection, for instance, if (5) denotes the stencil for the original matrix A , then the stencils (or equations) for the

matrices $L_1, L_2, L_3, L_1^T, L_2^T, L_3^T$, and M are given by

$$\begin{aligned}
\text{stencil for } M &= mu_{i,j,k}, \\
\text{stencil for } L_1 &= -l_1 u_{i-1,j,k}, \\
\text{stencil for } L_1^T &= -l_1 u_{i+1,j,k}, \\
\text{stencil for } L_2 &= -l_2 u_{i,j-1,k}, \\
\text{stencil for } L_2^T &= -l_2 u_{i,j+1,k}, \\
\text{stencil for } L_3 &= -l_3 u_{i,j,k-1}, \\
\text{stencil for } L_3^T &= -l_3 u_{i,j,k+1}.
\end{aligned} \tag{14}$$

We recall here that the matrices $M, L_1, L_2, L_3, L_1^T, L_2^T$, and L_3^T are all circulant matrices of same size as the original coefficient matrix A , thus they share the same set of eigenvectors given by (10). After substituting the eigenvector (10) in (14) for $u_{i,j,k}$, a straightforward computation gives the required eigenvalues in (13). The Fourier eigenvalues for the HSSOR preconditioner B for the isotropic case, $l_1 = l_2 = l_3 = 1$, is derived as follows

$$\begin{aligned}
\lambda_{s,t,r}(B) &= (\lambda_{s,t,r}(P) + \lambda_{s,t,r}(L_3)) \left(1 + \frac{\lambda_{s,t,r}(L_3^T)}{\lambda_{s,t,r}(P)} \right), \\
&= \lambda_{s,t,r}(P) + \lambda_{s,t,r}(L_3) + \lambda_{s,t,r}(L_3^T) + \frac{\lambda_{s,t,r}(L_3)\lambda_{s,t,r}(L_3^T)}{\lambda_{s,t,r}(P)}, \\
&= \lambda_{s,t,r}(P) - e^{\iota\xi_r} - e^{-\iota\xi_r} + \frac{e^{\iota\xi_r}e^{-\iota\xi_r}}{\lambda_{s,t,r}(P)} = \lambda_{s,t,r}(P) + \frac{1}{\lambda_{s,t,r}(P)} - 2\cos(\xi_r),
\end{aligned}$$

where $\lambda_{s,t,r}(P)$ is derived in a similar way as follows

$$\begin{aligned}
\lambda_{s,t,r}(P) &= (\lambda_{s,t,r}(T) + \lambda_{s,t,r}(L_2)) \left(1 + \frac{\lambda_{s,t,r}(L_2^T)}{\lambda_{s,t,r}(T)} \right), \\
&= \lambda_{s,t,r}(T) + \lambda_{s,t,r}(L_2) + \lambda_{s,t,r}(L_2^T) + \frac{\lambda_{s,t,r}(L_2)\lambda_{s,t,r}(L_2^T)}{\lambda_{s,t,r}(T)}, \\
&= \lambda_{s,t,r}(T) - e^{\iota\phi_t} - e^{-\iota\phi_t} + \frac{e^{\iota\phi_t}e^{-\iota\phi_t}}{\lambda_{s,t,r}(T)} = \lambda_{s,t,r}(T) + \frac{1}{\lambda_{s,t,r}(T)} - 2\cos(\phi_t),
\end{aligned}$$

where $\lambda_{s,t,r}(T) = 6 - 2\cos(\theta_s)$. Due to the periodicity of eigenvalues, i.e.,

$$\begin{aligned}
\lambda_{s,t,r}(A) |_{(\theta_s, \phi_t, \xi_r)} &= \lambda_{s,t,r}(A) |_{(2\pi-\theta_s, 2\pi-\phi_t, 2\pi-\xi_r)}, \\
\lambda_{s,t,r}(B) |_{(\theta_s, \phi_t, \xi_r)} &= \lambda_{s,t,r}(B) |_{(2\pi-\theta_s, 2\pi-\phi_t, 2\pi-\xi_r)},
\end{aligned}$$

we will restrict our domain to $(0, \pi)$ instead of $(0, 2\pi)$. For any arbitrary matrix K , we will use the notation $\lambda_{\min}(K)$ and $\lambda_{\max}(K)$ to denote the minimum and maximum eigenvalues respectively. When the expression $\lambda_{s,t,r}(K)$ does not depend on one or more of its arguments s, t , or r , in such case, we use the dummy argument “*”. For instance, $\lambda_{s,*,*}(K)$ is an expression independent of the arguments t and r .

Lemma 3. *If θ_s, ϕ_t , and ξ_r lie in the interval $(0, \pi)$, then following holds*

1. $\lambda_{\min}(A) = \lambda_{1,1,1}(A) > 0$,
2. $\lambda_{\min}(T) = \lambda_{1,*,*}(T) > 4$,
3. $\lambda_{\min}(P) = \lambda_{1,1,*}(P) > 9/4$,
4. $\lambda_{\min}(B) = \lambda_{1,1,1}(B) > 95/36$,
5. $\lambda_{\max}(T) = \lambda_{n/2,*,*}(T) < 8$,
6. $\lambda_{\max}(P) = \lambda_{n/2,n/2,*}(P) < 81/8$,
7. $\lambda_{\max}(B) = \lambda_{n/2,n/2,n/2}(B) < 7921/648$,

8. $\lambda_{max}(A) = \lambda_{1,1,1}(A) < 12$.

Proof. We observe that $\lambda_{min}(A) = 4(\sin^2(\theta_1/2) + \sin^2(\phi_1/2) + \sin^2(\xi_1/2)) = \lambda_{1,1,1}(A) > 0$. To prove other parts of the lemma, we have $\lambda_{min}(T) = 6 - 2\cos(\theta_1) = \lambda_{1,*,*}(T) > 4$. Now given $x > 1$, the expression $x + \frac{1}{x}$ increases or decreases according as x increases or decreases, consequently, we have

$$\begin{aligned}\lambda_{min}(P) &= \lambda_{min}(T) + \frac{1}{\lambda_{min}(T)} - \max(2\cos(\phi_t)), \\ &= \lambda_{1,*,*}(T) + \frac{1}{\lambda_{1,*,*}(T)} - \max(2\cos(\phi_1)), \\ &= \lambda_{1,1,*}(P) > 9/4.\end{aligned}$$

Similarly, we have

$$\lambda_{min}(B) = \lambda_{min}(P) + \frac{1}{\lambda_{min}(P)} - \max(2\cos(\xi_r)) = \lambda_{1,1,1}(B) > 95/36.$$

On the other hand for the upper bounds, we have $\lambda_{max}(T) = 6 - 2\cos(\theta_{n/2}) = \lambda_{n/2,*,*}(T) < 8$, and

$$\lambda_{max}(P) = \lambda_{max}(T) + \frac{1}{\lambda_{max}(T)} - \min(2\cos(\phi_t)) = \lambda_{n/2,n/2,*}(P) < 81/8.$$

Similarly, we have

$$\lambda_{max}(B) = \lambda_{max}(P) + \frac{1}{\lambda_{max}(P)} - \min(2\cos(\xi_r)) = \lambda_{n/2,n/2,n/2}(B) < 7921/648.$$

Finally, it is clear that $\lambda_{max}(A) = \lambda_{n/2,n/2,n/2}(A) < 12$. Hence, the lemma is proved. \square

In the following Lemma, we determine the eigenvalues of $B - A$.

Lemma 4. *The eigenvalues of $B - A$ are given as follows*

$$\lambda_{s,t,r}(B - A) = \frac{1}{\lambda_{s,t,r}(T)} + \frac{1}{\lambda_{s,t,r}(P)}.$$

Proof. We have

$$\begin{aligned}\lambda_{s,t,r}(B) &= \lambda_{s,t,r}(P) + \frac{1}{\lambda_{s,t,r}(P)} - 2\cos(\xi_r), \\ &= \lambda_{s,t,r}(T) + \frac{1}{\lambda_{s,t,r}(T)} + \frac{1}{\lambda_{s,t,r}(P)} - 2(\cos(\phi_t) + \cos(\xi_r)), \\ &= 6 - 2(\cos(\theta_t) + \cos(\phi_r) + \cos(\xi_r)) + \frac{1}{\lambda_{s,t,r}(T)} + \frac{1}{\lambda_{s,t,r}(P)}, \\ &= \lambda_{s,t,r}(A) + \frac{1}{\lambda_{s,t,r}(T)} + \frac{1}{\lambda_{s,t,r}(P)}.\end{aligned}$$

The proof is complete. \square

Following lemma will be useful in estimating condition number of HSSOR.

Lemma 5. *There holds*

$$\begin{aligned}\lambda_{max}(B - A) &= \lambda_{1,1,1}(B^{-1}A), \\ \lambda_{min}(B - A) &= \lambda_{n/2,n/2,n/2}(B^{-1}A).\end{aligned}$$

Proof. Using Lemma 4, we have

$$\lambda_{s,t,r}(B - A) = \lambda_{s,t,r}(B) - \lambda_{s,t,r}(A) = \frac{1}{\lambda_{s,t,r}(T)} + \frac{1}{\lambda_{s,t,r}(P)}, \quad (15)$$

and using Lemma 3 above, we have

$$\begin{aligned} \lambda_{max}(B - A) &= \frac{1}{\lambda_{min}(T)} + \frac{1}{\lambda_{min}(P)} = \frac{1}{\lambda_{1,*,*}(T)} + \frac{1}{\lambda_{1,1,*}(P)} = \lambda_{1,1,1}(B^{-1}A), \\ \lambda_{min}(B - A) &= \frac{1}{\lambda_{max}(T)} + \frac{1}{\lambda_{max}(P)} = \frac{1}{\lambda_{n/2,*,*}(T)} + \frac{1}{\lambda_{n/2,n/2,*}(P)} = \lambda_{n/2,n/2,n/2}(B^{-1}A). \end{aligned}$$

□

Lemma 6. *There holds*

$$\begin{aligned} \lambda_{max}(B^{-1}A) &= \lambda_{1,1,1}(B^{-1}A), \\ \lambda_{min}(B^{-1}A) &= \lambda_{n/2,n/2,n/2}(B^{-1}A). \end{aligned}$$

Proof. We have

$$\lambda_{s,t,r}(B^{-1}A) = \frac{\lambda_{s,t,r}(A)}{\lambda_{s,t,r}(B)} = \frac{1}{1 + \lambda_{s,t,r}^{-1}(A)(\lambda_{s,t,r}(B - A))}.$$

From Lemma 3 above, we have $\lambda_{s,t,r}^{-1}(A) > 0$ and $\lambda_{s,t,r}(B - A) > 0$, thus, using Lemma 5 we can write the following

$$\lambda_{min}(B^{-1}A) = \frac{1}{1 + \lambda_{min}^{-1}(A)(\lambda_{max}(B - A))} = \lambda_{1,1,1}(B^{-1}A), \quad (16)$$

$$\lambda_{max}(B^{-1}A) = \frac{1}{1 + \lambda_{max}^{-1}(A)(\lambda_{min}(B - A))} = \lambda_{n/2,n/2,n/2}(B^{-1}A). \quad (17)$$

The proof is complete. □

The matrix $B^{-1}A$ is symmetric by construction, the following corollary proves that it is also positive definite.

Corollary 1. *The HSSOR preconditioned matrix $B^{-1}A$ is symmetric positive definite.*

Proof. Using Lemma 6 and Lemma 3 above, we have

$$\lambda_{min}(B^{-1}A) = \lambda_{n/2,n/2,n/2}(B^{-1}A) = \lambda_{s,t,r}(A)/\lambda_{s,t,r}(B) > 0.$$

The proof is complete. □

The following theorem gives a condition number estimate of HSSOR preconditioned matrix. Let $\text{cond}(K)$ for any matrix K denote the condition number of K . The notation \approx will mean ‘approximately equal to’.

Theorem 1. *Let h tends to zero, then the condition number of the HSSOR preconditioned matrix $B^{-1}A$ is given as follows*

$$\text{cond}(B^{-1}A) \approx \left(\frac{25(5 + 5\pi^2 + \pi^4)}{144(3\pi^2(5 + 5\pi^2 + \pi^4) + 4\pi^2)} \right) h^{-2}.$$

Proof. From Lemma 6 above, the condition number of HSSOR preconditioned matrix, $\text{cond}(B^{-1}A)$, is given as follows

$$\text{cond}(B^{-1}A) = \frac{\lambda_{\max}(B^{-1}A)}{\lambda_{\min}(B^{-1}A)} = \frac{\lambda_{n/2,n/2,n/2}(B^{-1}A)}{\lambda_{1,1,1}(B^{-1}A)}.$$

Using Lemma 3 and recalling that $\theta_s = 2\pi s/(n+1) = (2\pi h)s$, We have $\cos(\theta_1) = \cos(2\pi h) \approx 1 - 2\pi^2 h^2$, consequently, we have $\lambda_{1,*,*}(T) = 6 - 2\cos(\theta_1) \approx 4(1 + \pi^2 h^2)$, thus using Lemma 4 and $\cos(\phi_1) = 1 - 2\pi^2 h^2$, we have

$$\begin{aligned} \lambda_{\max}(B - A) &= \lambda_{1,1,1}(B - A), \\ &= \frac{1}{\lambda_{1,*,*}(T)} + \frac{1}{\lambda_{1,*,*}(T) + \lambda_{1,*,*}^{-1}(T) - 2\cos(\phi_1)}, \\ &\approx \frac{1}{4(1 + \pi^2 h^2)} + \frac{1}{4(1 + \pi^2 h^2) + (1/4)(1 + \pi^2 h^2)^{-1} - 2(1 - 2\pi^2 h^2)}, \\ &\approx \frac{1}{4(1 + \pi^2 h^2)} + \frac{1}{4(1 + \pi^2 h^2) + (1/4)(1 - \pi^2 h^2) - 2(1 - 2\pi^2 h^2)}, \\ &= \frac{1}{4(1 + \pi^2 h^2)} + \frac{1}{2 + 1/4 + (8 - 1/4)\pi^2 h^2}, \\ &= \frac{1}{4(1 + \pi^2 h^2)} + \frac{4}{9 + 31\pi^2 h^2} = \frac{25 + 47\pi^2 h^2}{4(1 + \pi^2 h^2)(9 + 31\pi^2 h^2)}. \end{aligned}$$

we have $\lambda_{\min}(A) = \lambda_{1,1,1}(A) = 4(\sin^2(\theta_1/2 + \phi_1/2 + \xi_1/2)) \approx 12\pi^2 h^2$. From above estimates and from expression (16) we have

$$\begin{aligned} \lambda_{\min}(B^{-1}A) &= \lambda_{n/2,n/2,n/2}(B^{-1}A), \\ &= \frac{1}{1 + \lambda_{\min}^{-1}(A)(\lambda_{\max}(B - A))}, \\ &= \frac{1}{1 + (1/12\pi^2 h^2)(25 + 47\pi^2 h^2)/(4(1 + \pi^2 h^2)(9 + 31\pi^2 h^2))}, \\ &= \frac{12\pi^2 h^2}{12\pi^2 h^2 + (25 + 47\pi^2 h^2)/(4(1 + \pi^2 h^2)(9 + 31\pi^2 h^2))}, \\ &= \frac{12\pi^2 h^2 \cdot 4(1 + \pi^2 h^2)(9 + 31\pi^2 h^2)}{4 \cdot 12\pi^2 h^2(1 + \pi^2 h^2)(9 + 31\pi^2 h^2) + 25 + 47\pi^2 h^2}, \\ &= \frac{4 \cdot 9 \cdot 12\pi^2 h^2 + O(h^4)}{25 + O(h^2)} \approx \frac{4 \cdot 9 \cdot 12\pi^2 h^2}{25}. \end{aligned}$$

We recall that $\theta_s = 2\pi s/(n+1)$. For $s = n/2$, we have $\theta_{n/2} = n\pi/(n+1) = \pi - \pi/(n+1) = (1 - h)\pi$ (since, $1/(n+1) = h$). We have $\cos(\theta_{n/2}) \approx 1 - (\theta_{n/2}^2)/2 = 1 - (1 - h)^2 \pi^2 / 2 \approx ((2 - \pi^2) + 2\pi^2 h)/2$. Due to symmetry, we have $\cos(\phi_{n/2}) = \cos(\xi_{n/2}) = \cos(\theta_{n/2})$. Consequently, $\lambda_{n/2,*,*}(T) = 6 - 2\cos(\theta_{n/2}) = 4 + \pi^2 - 2\pi^2 h$. Using Lemma 4, we have

$$\begin{aligned} \lambda_{\min}(B - A) &= \lambda_{n/2,n/2,n/2}(B - A), \\ &= \frac{1}{\lambda_{n/2,*,*}(T)} + \frac{1}{\lambda_{n/2,*,*}(T) + \lambda_{n/2,*,*}^{-1}(T) - 2\cos(\phi_{n/2})}, \\ &\approx \frac{1}{4 + \pi^2 - 2\pi^2 h} + \frac{1}{(4 + \pi^2 - 2\pi^2 h) + (4 + \pi^2 - 2\pi^2 h)^{-1} - ((2 - \pi^2) + 2\pi^2 h)/2}, \\ &\approx \frac{1}{4 + \pi^2 - 2\pi^2 h} + \frac{1}{(4 + \pi^2 - 2\pi^2 h) + (4 + \pi^2)^{-1}(1 + 2\pi^2 h/(4 + \pi^2)) - ((2 - \pi^2) + 2\pi^2 h)/2}, \\ &= \frac{1}{4 + \pi^2 - 2\pi^2 h} + \frac{1}{4 + \pi^2 + 1/(4 + \pi^2) - (2 - \pi^2)/2 + O(h)}, \\ &= \frac{2(4 + \pi^2) + O(h)}{(4 + \pi^2) + 2 - (2 - \pi^2)(4 + \pi^2) + O(h)} \approx \frac{4 + \pi^2}{5 + 5\pi^2 + \pi^4}. \end{aligned}$$

From Lemma 3, we have $\lambda_{max}(A) = \lambda_{n/2, n/2, n/2}(A) = 6(1 - \cos \theta_{n/2}) \approx 6 - 3(2 - \pi^2 + 2\pi^2 h) = 3\pi^2(1 - 2h)$. Thus, we have

$$\begin{aligned}\lambda_{max}(B^{-1}A) &= \frac{1}{1 + \lambda_{max}^{-1}(A)(\lambda_{min}(B - A))}, \\ &\approx \frac{1}{1 + (1/(3\pi^2(1 - 2h))((4 + \pi^2)/(5 + 5\pi^2 + \pi^4)))}, \\ &= \frac{3\pi^2(1 - 2h)(5 + 5\pi^2 + \pi^4)}{3\pi^2(1 - 2h)(5 + 5\pi^2 + \pi^4) + 4 + \pi^2}, \\ &= \frac{3\pi^2(5 + 5\pi^2 + \pi^4) + O(h)}{3\pi^2(5 + 5\pi^2 + \pi^4) + 4 + \pi^2 + O(h)}.\end{aligned}$$

The condition number of $B^{-1}A$ is given as follows

$$\begin{aligned}\text{cond}(B^{-1}A) &= \frac{3\pi^2(5 + 5\pi^2 + \pi^4) + O(h)}{3\pi^2(5 + 5\pi^2 + \pi^4) + 4 + \pi^2 + O(h)} \times \frac{25}{4 \cdot 9 \cdot 12\pi^2 h^2}, \\ &\approx \left(\frac{25(5 + 5\pi^2 + \pi^4)}{144(3\pi^2(5 + 5\pi^2 + \pi^4) + 4\pi^2)} \right) h^{-2} \approx (0.006)h^{-2}.\end{aligned}$$

The proof is complete. \square

4 Two grid scheme

In classical AMG, a set of coarse grid unknowns is selected and the matrix entries are used to build interpolation rules that define the prolongation matrix P , and the coarse grid matrix A_c is computed from the following Galerkin formula

$$A_c = P^T A P. \quad (18)$$

In contrast to the classical AMG approach, in aggregation based multigrid, first a set of aggregates G_i are defined. Let N_c be the number of such aggregates, then the interpolation matrix P is defined as follows

$$P_{ij} = \begin{cases} 1, & \text{if } i \in G_j, \\ 0, & \text{otherwise,} \end{cases}$$

Here, $1 \leq i \leq N$, $1 \leq j \leq N_c$, N being the size of the original coefficient matrix A . Further, we assume that the aggregates G_i are such that

$$G_i \cap G_j = \emptyset, \text{ for } i \neq j \text{ and } \cup_i G_i = [1, N] \quad (19)$$

Here $[1, N]$ denotes the set of integers from 1 to N . Notice that the matrix P defined above is an $N \times N_c$ matrix, but since it has only one non-zero entry (which are “one”) per row, the matrix can be defined by a single array containing the indices of the non-zero entries. The coarse grid matrix A_c may be computed as follows

$$(A_c)_{ij} = \sum_{k \in G_i} \sum_{l \in G_j} a_{kl}$$

where $1 \leq i, j \leq N_c$, and a_{kl} is the (k, l) th entry of A .

Numerous aggregation schemes have been proposed in the literature, but in this paper we consider the aggregation scheme based on graph matching as follows

Aggregation based on graph matching: Several graph partitioning methods exists, notably, in software form [5]. Aggregation for AMG is created by calling a graph partitioner with required number of aggregates as an input. The subgraph being partitioned are the aggregates G_i . For instance, in this paper we

use this approach by calling the METIS graph partitioning routine, namely, METIS_PartGraphKway with the graph of the matrix and number of partitions as input parameters. The partitioning information is obtained in the output argument “part”. The part array maps a given node to its partition, i.e., $\text{part}(i) = j$ means that the node i is mapped to the j th partition. In fact, the part array essentially determines the interpolation operator P . For instance, we observe that the “part” array is a discrete many to one map. Thus, the i th aggregate $G_i = \text{part}^{-1}(i)$, where

$$\text{part}^{-1}(i) = \{j \in [1, N] \mid \text{part}(j) = i\}$$

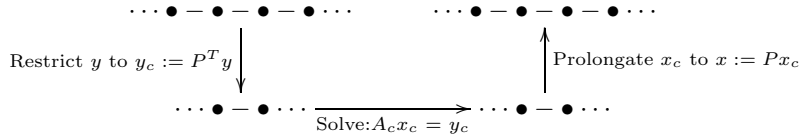
Such graph matching techniques for AMG coarsening were also explored in [6, 11]. For notational convenience, the method introduced in this paper is called GMG (Graph matching based aggregation MultiGrid).

Let S denote the matrix which acts as a smoother in GMG method. The usual choice of S is a Gauss-Siedel or SSOR preconditioner [12]. However, in this paper we choose HSSOR as a smoother and compare it with SSOR.

Let $M = PA_cP^T$ denote the coarse grid operator *interpolated* to fine grid, then the two-grid preconditioner without post-smoothing is defined as follows

$$B = (S^{-1} + M^{-1} - M^{-1}AS^{-1})^{-1}. \quad (20)$$

We notice that $M^{-1} \approx PA_c^{-1}P^T$, thus, an equation of the form $Mx = y$ is solved by first restricting y to $y_c = P^Ty$, then solving with the coarse matrix A_c the following linear system: $A_cx_c = y_c$. Finally, prolongating the coarse grid solution x_c to $x = Px_c$. Following diagram illustrates the two-grid hierarchy.



5 Numerical experiments

All the numerical experiments were performed in MATLAB with double precision accuracy on Intel core i7 (720QM) with 6 GB RAM. The AMG method introduced in this paper, namely, GMG, is written in MATLAB. For GMG, the iterative accelerator used is GMRES available at [13], the code was changed such that the stopping is based on the decrease of the 2-norm of the relative residual. For both GMRES, the maximum number of iterations allowed is 500, and the inner subspace dimension is 30. The stopping criteria is the decrease of the relative residual below 10^{-10} , i.e., when

$$\frac{\|b - Ax_k\|}{\|b\|} < 10^{-10}.$$

Here b is the right hand side and x_k is an approximation to the solution at the k th step.

5.1 Test cases

Diffusion: Our primary test case is the following diffusion Equation. We use the notation DC to indicate that the problems are discontinuous. We consider a test case as follows

$$\begin{aligned} -\text{div}(\kappa(x)\nabla u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega_D, \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega_N, \end{aligned} \quad (21)$$

Table 1: Notations used in tables of numerical experiments

Notations	Meaning
h	Discretization step
N	Size of the original matrix
N_c	Size of the coarse grid matrix
its	Iteration count
time	Total CPU time (setup plus solve) in seconds
cf	$(N)^{1/3}/(N_c)^{1/3}$
ME	Memory allocation problem
NA	Not applicable
NC	Not converged within 500 iterations
SSOR	Symmetric successive over-relaxation method ($\omega = 1$)[12]
BSSOR	Block symmetric successive over-relaxation method ($\omega = 1$)[12]
HSSOR	Hierarchical SSOR
GMG-HS	Graph based matching for AMG, smoother HSSOR
GMG-SS	Graph based matching for AMG, smoother SSOR

DC1, 2D case: The tensor κ is isotropic and discontinuous. The domain contains many zones of high permeability that are isolated from each other. Let $[x]$ denote the integer value of x . For two-dimensional case, we define $\kappa(x)$ as follows:

$$\kappa(x) = \begin{cases} 10^3 * ([10 * x_2] + 1), & \text{if } [10 * x_i] \equiv 0 \pmod{2}, i = 1, 2, \\ 1, & \text{otherwise.} \end{cases}$$

The velocity field \mathbf{a} is kept zero. We consider a $n \times n$ uniform grid where n is the number of discrete points along each spatial directions.

DC1, 3D case: For three-dimensional case, $\kappa(x)$ is defined as follows:

$$\kappa(x) = \begin{cases} 10^3 * ([10 * x_2] + 1), & \text{if } [10 * x_i] \equiv 0 \pmod{2}, i = 1, 2, 3, \\ 1, & \text{otherwise.} \end{cases}$$

Here again, the velocity field \mathbf{a} is kept zero. We consider a $n \times n \times n$ uniform grid.

Isotropic problem: i.e., we have $l1 = l2 = l3 = 1$ and $d = 6$ for the model problem (21). More precisely, the matrix is given as follows

$$A = Btrid_n(-l_3 I_{n^2}, \widehat{D}, -l_3 I_{n^2}), \quad \widehat{D} = Btrid_n(-l_2 I_n, \overline{D}, -l_2 I_n), \quad \overline{D} = trid_n(-l_1, d, -l_1).$$

For the 2D case, we have $l1 = l2 = 1$, and $d = 4$, the matrix after discretization is given as follows

$$A = Btrid_n(-l_2 I_n, \overline{D}, -l_2 I_n), \quad \overline{D} = trid_n(-l_1, d, -l_1).$$

Comments on numerical experiments

In Table 2, we compare the two-grid methods, ILU(0), HSSOR, SSOR, and BSSOR. In two grid methods, GMG-SS has point SSOR smoother while GMG-HS has HSSOR as a smoother. For the 2D case and for $1/h \geq 400$, the ILU(0), SSOR, HSSOR, and BSSOR does not converge within 500 iterations, while all of them converge for 3D problem except for large size problem where insufficient memory error occurs. In particular, for block SSOR method, we need to store the LU factors corresponding to the 2D blocks. This

Table 2: Numerical results for isotropic model problem with $cf = 4.5$ using GMRES(30), maximum number of iterations allowed is 500

matrix	$1/h$	GMG-HS		GMG-SS		ILU(0)		HSSOR		SSOR		BSSOR	
		its	time	its	time	its	time	its	time	its	time	its	time
2D	400	39	6.2	46	6.8	NC	NA	NC	NA	NC	NA	NC	NA
	800	39	28.9	47	37.1	NC	NA	NC	NA	NC	NA	NC	NA
	1000	42	47.5	50	60.9	NC	NA	NC	NA	NC	NA	NC	NA
3D	40	23	1.3	21	3.7	55	1.8	42	1.6	68	2.7	127	83.4
	80	25	24.5	22	138.7	129	61.9	89	38.7	157	76.6	ME	NA
	100	26	72.3	ME	NA	147	138.3	113	106.4	185	180.7	ME	NA

Table 3: Numerical results for DC1 problem with $cf = 3$ using GMRES(30). Maximum number of iterations allowed is 500

matrix	$1/h$	GMG-HS		GMG-SS		ILU(0)		HSSOR		SSOR		BSSOR	
		its	time	its	time	its	time	its	time	its	time	its	time
2D	400	29	5.7	35	10.5	NC	NA	NC	NA	NC	NA	NC	NA
	800	29	33.4	34	49.5	NC	NA	NC	NA	NC	NA	NC	NA
	1000	29	58.4	35	85.0	NC	NA	NC	NA	NC	NA	NC	NA
3D	40	247	18.5	300	44.0	475	14.2	NC	NA	NC	NA	NC	NA
	80	237	337.5	281	895.9	NC	NA	NC	NA	NC	NA	NC	NA
	100	ME	NA	ME	NA	NC	NA	NC	NA	NC	NA	NC	NA

is the reason why we refrain from using BSSOR as a smoother in the two-grid scheme. The iteration count as well as CPU time for HSSOR is smaller compared to ILU(0) and the difference between iteration count and CPU time increases with the size of the problem.

In Table 2, as expected, the two-grid methods show mesh independent convergence, and the CPU time and iteration count is much less than that of the stand alone methods. Comparing the two-grid versions GMG-SS and GMG-HS, we find that GMG-HS is an improvement over GMG-SS, particularly, for the 2D isotropic problem.

In Table 3, we show experimental results for a discontinuous DC1 problem both for 2D and 3D problem. This problem is difficult compared to isotropic case, we had to keep a smaller cf value of 3. For $cf = 4.5$, the two-grid methods did not converge within 500. Notably, neither of the stand-alone methods converged within 500 iterations. However, for $cf = 3$, the two-grid method shows mesh independent convergence with GMG-HS taking relatively less iterations, and takes less CPU time compared to GMG-SS.

6 Conclusion

In this paper, we have obtained a condition number estimate for a hierarchical SSOR method. The estimate facilitates comparison with the condition number of ILU(0) obtained in [4]. Numerical experiments shows that the HSSOR is faster compared to ILU(0), SSOR, and BSSOR as a stand alone preconditioner. Moreover, for a two-grid method, we show that HSSOR is an efficient smoother and thus could replace the widely used Gauss-Siedel or SSOR smoother.

References

- [1] Appleyard, J. R., Cheshire, I. M.: Nested Factorization, SPE 12264, presented at the seventh SPE symposium on reservoir simulation, San Francisco, (1983)
- [2] Chan, T. F., Elman, H. C.: Fourier analysis of iterative methods for elliptic boundary value problems. SIAM Review. **31**(1), 20-49 (1989)
- [3] Davis, P. J.: Circulant Matrices, 2nd ed. New York, Chelsea, (1994)
- [4] Donato, J. M., Chan, T. F.: Fourier analysis of incomplete factorization preconditioners for three dimensional anisotropic problems. SIAM J. Sci. Stat. Comput. **13**(1), 319-338 (1992)
- [5] Karypis, G., Kumar, V.: A fast and high quality multilevel scheme for partitioning irregular graphs, SIAM J. Sci. Comput. **20**(1), 359-392 (1999)
- [6] Kim, H., Xu, J., Zikatanov, L.: A multigrid method based on graph matching for convection-diffusion equations. Numer. Linear Algebra Appl.. **10**(1-2), 181-195 (2003)
- [7] Kumar, P., Grigori, L., Nataf, F., Niu, Q.: Combinative preconditioning based on relaxed nested factorization and tangential filtering preconditioner, INRIA Tech. report RR-6955
- [8] Niu, Q., Grigori, L., Kumar, P., Nataf, F.: Modified tangential filtering decomposition and its Fourier analysis. Numerische Mathematik. **116**(1), 123-148 (2010)
- [9] Notay, Y.: An aggregation based algebraic multigrid method. ETNA, Electron. Trans. Numer. Anal. **37**, 123-146 (2010)
- [10] Otto, K.: Analysis of Preconditioners for Hyperbolic Partial Differential Equations. SIAM J. Numer. Anal. **33**(6), 2131-2165 (1996)
- [11] Rasquin, M., Deconinck, H., Degrez, G.: FLEXMG: A new library of multigrid preconditioners for a spectral/finite element incompressible flow solver. Int. J. Numer. Methods Eng. **82**(12), 1510-1536 (2010)
- [12] Saad, Y.: Iterative Methods for Sparse Linear Systems, PWS publishing company, Boston, MA, (1996)
- [13] Saad, Y.: Matlab suite, Available online at: <http://www-users.cs.umn.edu/~saad/software/>
- [14] Wienands, R., Joppich, W.: Practical Fourier analysis for multigrid methods, Taylor and Francis Inc. (2004)